

AN INEQUALITY FOR MIXED MONGE-AMPÈRE MEASURES

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ABSTRACT. We generalize an inequality for mixed Monge-Ampère measures from [K1]. We also give an example that shows that our assumptions are sharp. The corresponding result in the setting of compact Kähler manifold is also discussed.

1. INTRODUCTION

The complex Monge-Ampère operator acts on a smooth plurisubharmonic function u defined in \mathbb{C}^n by

$$MA(u) := 4^n n! \det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right) d\lambda = (dd^c u)^n.$$

For this reason the theory of positive definite matrices has found numerous applications in the study of this operator. We refer to [HJ] for the basic ideas in this vein. For example one can use concavity properties of such matrices to obtain pointwise estimates for the Monge-Ampère operator (see [W1]). In particular the following holds:

Theorem 1.1. *Let u, v be bounded and smooth plurisubharmonic functions such that $(dd^c u)^n \geq f d\lambda$, $(dd^c v)^n \geq g d\lambda$, where f and g are smooth nonnegative functions and $d\lambda$ is the Lebesgue measure. Then*

$$(1.1) \quad (dd^c u)^k \wedge (dd^c v)^{n-k} \geq f^{\frac{k}{n}} g^{\frac{n-k}{n}} d\lambda$$

$$(1.2) \quad (dd^c(u+v))^n \geq (f^{\frac{1}{n}} + g^{\frac{1}{n}})^n d\lambda$$

Those inequalities were first used in [BT1] in order to generalize the action of MA to nonsmooth functions. Very similar estimates were also used by Demailly in problems in complex geometry (see [De]). Last but not least, such inequalities were used in [K1] together with the comparison principle to produce stability estimates for the complex Monge-Ampère operator.

Of course plurisubharmonic functions are, in general, neither smooth nor even bounded. However the Monge-Ampère operator can still be reasonably defined (in the weak sense) for quite general functions (see [BT1], [BT2], [Bl1], [Bl2], [Ce1], [Ce2]). This leads to natural questions, namely whether our inequalities hold in this more general situation. Note also that pluripotential theory deals often with measures singular w.r.t. the Lebesgue measure. An example of such a situation arises when one computes the Monge-Ampère of a maximum of two smooth plurisubharmonic functions. Measures associated to maximal plurisubharmonic functions provide another example. Hence one can pose a question in the following way:

Question: *Let u, v be plurisubharmonic functions (psh for short) such that their Monge-Ampère masses $(dd^c u)^n, (dd^c v)^n$ are well defined (in the sense of [Bl2], or equivalently in the sense of [Ce2]). Let also μ be a positive measure and $f, g \in L^1(d\mu)$. Suppose*

that $(dd^c u)^n \geq f d\mu$, $(dd^c v)^n \geq g d\mu$ in the sense of measures. Is it true that (again in the sense of measures)

$$(1.3) \quad (dd^c u)^k \wedge (dd^c v)^{n-k} \geq f^{\frac{k}{n}} g^{\frac{n-k}{n}} d\mu$$

$$(1.4) \quad (dd^c(u+v))^n \geq (f^{\frac{1}{n}} + g^{\frac{1}{n}})^n d\mu$$

Some results in this direction are known. Inequalities (1.3) and (1.4) were generalized to nonsmooth u and v , but with the Lebesgue measure instead of $d\mu$ (see [K1], [B13]). In particular the following holds (see [K1]):

Theorem 1.2. *Assume that u and v are plurisubharmonic and bounded functions in a domain in \mathbb{C}^n . Let also $f, g \in L^1(d\lambda)$ are nonnegative functions such that*

$$(dd^c u)^n \geq f d\lambda, \quad (dd^c v)^n \geq g d\lambda$$

holds. Then

$$(dd^c u)^k \wedge (dd^c v)^{n-k} \geq f^{\frac{k}{n}} g^{\frac{n-k}{n}} d\lambda, \quad k = 1, \dots, n$$

Remark. In [K1] this was stated for continuous u and v , but the proof applies without changes to the more general case. The crucial thing in this result is the Lebesgue measure on the right hand side: this enables us to find a nice approximation sequence of smooth functions and use the smooth version to prove the result.

The aim of this note is to study these inequalities for more general functions. In particular we show that they, in general, do not hold. We also give sharp condition for the measure μ ensuring the positive answer. It is very interesting that this condition (vanishing on pluripolar sets) is closely related to uniqueness for the Dirichlet problem for the Monge-Ampère operator. Also the same condition appears when one generalizes the Demailly inequality

$$(1.5) \quad (dd^c \max(u, v))^n \geq \chi_{u \geq v} (dd^c u)^n + \chi_{u < v} (dd^c v)^n$$

(see [KH]). Let us state our main result:

Theorem 1.3. *Let μ be a positive measure on Ω that vanishes on all pluripolar sets. Let $u_1, u_2, \dots, u_n \in PSH(\Omega)$ be plurisubharmonic functions with well defined Monge-Ampère operator (we can work on a fixed domain Ω as in [Ce1], or, instead, we can work with germs of functions as in [B12]). Let also f_i , $i = 1, \dots, n$ be nonnegative functions integrable with respect to μ . If*

$$(dd^c u_i)^n \geq f_i d\mu, \quad \forall i = 1, \dots, n$$

then

$$dd^c u_1 \wedge dd^c u_2 \wedge \dots \wedge dd^c u_n \geq (f_1 f_2 \dots f_n)^{\frac{1}{n}} d\mu$$

Remark. To unify the two possible approaches we shall work locally (in a small ball). It will be explained what happens if we have a fixed domain Ω and all objects are globally defined there. Also, for the sake of brevity, we shall work throughout the note with two functions u and v instead of the collection of n functions. It will be explained how to get this general case.

Acknowledgement. I would like to thank my advisor professor Sławomir Kołodziej for very stimulating discussions.

2. DEFINITIONS

We shall work mostly in \mathbb{C}^n but, since we prove purely local results, everything can be generalized to manifold setting. The corresponding result in the setting of compact Kähler manifolds will be discussed in the last section.

As usual $d = \partial + \bar{\partial}$ and $d^c := i(\bar{\partial} - \partial)$, so $dd^c = 2i\partial\bar{\partial}$. The complex Monge-Ampère operator is defined to be

$$(dd^c u)^n := \underbrace{dd^c u \wedge \cdots \wedge dd^c u}_{n\text{-times}}$$

This definition at first makes sense only for smooth functions (since we cannot multiply distributions), but classical results in pluripotential theory allow to define this also for bounded plurisubharmonic functions.

A domain $\Omega \subset \mathbb{C}^n$ is called *hyperconvex* if it admits a negative plurisubharmonic exhaustion function i.e.

$$u \in PSH(\Omega), \quad u < 0, \quad \forall c > 0 \quad \{z \in \Omega \mid u(z) < -c\} \Subset \Omega$$

A smoothly bounded pseudoconvex domain is called *strictly pseudoconvex* if it admits a defining function that is strictly plurisubharmonic in $\bar{\Omega}$.

Let

$$\mathcal{E}_0(\Omega) = \{u \in PSH(\Omega) \cap L^\infty(\Omega) \mid u < 0, \lim_{z \rightarrow \zeta} u(z) = 0, \quad \forall \zeta \in \partial \Omega, \quad \int_{\Omega} (dd^c u)^n \leq \infty\}$$

denote the class of bounded exhaustion functions in a hyperconvex domain. Whenever it is clear with which domain we deal we shall write simply \mathcal{E}_0 instead of $\mathcal{E}_0(\Omega)$.

Let \mathcal{E}^p be the class of plurisubharmonic functions consisting of those functions g , for which there exists a sequence $g_j \in \mathcal{E}_0$ decreasing towards g , such that :

$$\sup_j \int_{\Omega} (-g_j)^p (dd^c g_j)^n < \infty.$$

Let $u \in \mathcal{E}^1$. The quantity

$$\int_{\Omega} -u (dd^c u)^n$$

is called \mathcal{E}^1 -norm of u . This is not a norm in the classical sense, nevertheless turns out to be a useful tool, as the following result (taken from [Cel]) shows:

Proposition 2.1. *Let $u, v \in \mathcal{E}^1(\Omega)$, $u \leq v$. Then*

$$\int_{\Omega} -v (dd^c v)^n \leq \int_{\Omega} -u (dd^c u)^n.$$

So \mathcal{E}^1 -norm of a function is controlled by the \mathcal{E}^1 -norm of a smaller function.

Now we recall more general Cegrell classes:

Let $\mathcal{F}(\Omega)$ be the class consisting of those functions g , for which there exists a decreasing sequence $g_j \in \mathcal{E}_0(\Omega)$ such that :

$$\sup_j \int_{\Omega} (dd^c g_j)^n < \infty.$$

Finally let \mathcal{E} be the class of functions g which locally belong to \mathcal{F} , i.e.

$$\forall w \in \Omega \quad \exists U_w \subset \Omega, \quad U_w \text{ -- open } \exists g_w \in \mathcal{F}(\Omega) : g|_{U_w} = g_w|_{U_w}.$$

These classes are usually called Cegrell classes (with boundary value 0) in the literature. Their main applications come from the fact that one can define uniquely their Monge-Ampère measure in such a way, that it is still continuous for decreasing sequences and moreover the definition is consistent with the classical one for smooth functions (which are basic properties for bounded plurisubharmonic functions).

There are very useful inequalities of Hölder type (due to Cegrell [Ce2]) in the class \mathcal{F} . They show how to control integrals of mixed Monge-Ampère type from above, in terms of \mathcal{E}^1 norms:

$$\begin{aligned} \int_{\Omega} -u(dd^c v_1) \wedge (dd^c v_2) \wedge \cdots \wedge (dd^c v_n) &\leq \left(\int_{\Omega} -u(dd^c v_1)^n \right)^{\frac{1}{n}} \cdots \left(\int_{\Omega} -u(dd^c v_n)^n \right)^{\frac{1}{n}}; \\ \int_{\Omega} -u(dd^c v_1) \wedge (dd^c v_2) \wedge \cdots \wedge (dd^c v_n) &\leq \\ &\leq \left(\int_{\Omega} -v_1(dd^c v_1)^n \right)^{\frac{1}{n+1}} \cdots \left(\int_{\Omega} -v_n(dd^c v_n)^n \right)^{\frac{1}{n+1}} \left(\int_{\Omega} -u(dd^c u)^n \right)^{\frac{1}{n+1}} \\ u &\in \mathcal{E}_0(\Omega), \quad v_1, \dots, v_n \in \mathcal{F}(\Omega). \end{aligned}$$

For more details concerning these topics we refer to [Ce1],[Ce2].

While we shall deal mostly with bounded plurisubharmonic functions in this note, we introduce the terminology of Cegrell classes, since it helps to simplify some proofs and to make other more transparent.

Given a positive Radon measure on a bounded domain $\Omega \in \mathbb{C}^n$ we define its *canonical approximation* (see [K4]): Let $\text{supp } \mu$ be contained in a big cube I . Consider a subdivision \mathcal{B}_k of I into 3^{2kn} congruent semi open cubes I_k^j , $j = 1, \dots, 3^{2kn}$. It is no loss of generality to assume $\mu(\cup_{I_k^j \in \mathcal{B}_k} \partial(I_k^j)) = 0$ (otherwise we can shift at each stage the boundaries a bit). Now define

$$(2.1) \quad \mu_k := \sum_j \frac{\mu(I_k^j \cap \Omega)}{dV(I_k^j \cap \Omega)} \chi_{I_k^j} dV,$$

where $\chi_{I_k^j}$ is the characteristic function of I_k^j . Of course μ_k is weak* convergent to μ and every term μ_k has a density in L^∞ with respect to the Lebesgue measure.

Below we list some results that we shall need later on:

Theorem 2.2. *Let Ω be a smoothly bounded strictly pseudoconvex domain in \mathbb{C}^n and let $f \in \mathcal{C}^\infty(\partial \Omega)$ be arbitrary. Let also μ be a positive measure on Ω with finite mass and compact support. Suppose μ satisfies the following condition for any $p > \frac{n}{n-1}$:*

There is a constant $A = A(p)$ such that

$$\int_{\Omega} (-\phi)^p d\mu \leq A \left(\int_{\Omega} (-\phi)^p (dd^c \phi)^n \right)^{\frac{p}{n+p}}$$

for any $\phi \in \mathcal{E}_0$. Then:

- (1) *There exist $u_k \in \mathcal{C}(\overline{\Omega})$ which solve the Dirichlet problem:*

$$\begin{cases} u_k \in PSH(\Omega) \cap C(\overline{\Omega}) \\ (dd^c u_k)^n = \mu_k \\ u_k = f \text{ on } \partial \Omega \end{cases}$$

where μ_k are the canonical approximants of μ .

- (2) *Define $u := (\limsup_{k \rightarrow \infty} u_k)^*$. Then there is a subsequence of $\{u_k\}$ (which after renumbering we still denote by $\{u_k\}$) such that $u_k \rightarrow u$ in $L^1(d\lambda)$.*

(3) We have for this sequence that

$$\sup_k \int_{\Omega} |-u_k|(dd^c u_k)^n < \infty$$

$$\lim_{k \rightarrow \infty} \int_{\Omega} |u - u_k|(dd^c u_k)^n = 0.$$

The proof of the first part may be found in [K4]. Other results follow from Theorems 5.1 and 7.7 and Lemmas 5.2, 5.3, 7.8 and 7.9 from [Ce1]. We would like to mention that the condition that all the functions have the same boundary values can be weakened. If, for example, the boundary values of u_k form a sequence decreasing towards a bounded upper semicontinuous function (which will be the case we shall use later on) $u := (\limsup_{k \rightarrow \infty} u_k)^*$ still makes perfect sense and, applying line by line the proofs from [Ce1] we get the finiteness and the convergence of the integrals in this situation too.

Theorem 2.3. *Suppose $u_j \in PSH(\Omega) \cap C(\overline{\Omega})$ is a sequence that converges to $u \in PSH(\Omega)$ in $L^1(d\lambda, \Omega)$. Suppose also all u_k (and hence also u) have the same continuous boundary values, i.e. $\lim_{z \rightarrow \zeta} u_j(z) = f(\zeta) \forall \zeta \in \partial \Omega$. If moreover $\lim_{k \rightarrow \infty} \int_{\Omega} |u - u_k|(dd^c u_k)^n = 0$ then u_k converges to u in capacity.*

This result is contained in the proof of Lemma 2.1 in [CK2]. Again we can carry the argument from [CK2] also if we let boundary values of u_j to decrease (to be precise, since in the proof there boundary values are used only to ensure relative compactness of sets $\{u_j < u - a\}$, $a > 0$, it is even better when boundary values of u_j are bigger than those of u).

Below we recall a definition of the envelope of subsolutions:

Definition 2.4. *Let μ be a positive Radon measure defined in a hyperconvex domain Ω and $f \in \mathcal{C}(\Omega)$. Then*

$$U(\mu, f)(z) := \sup\{v(z) \mid v \in PSH \cap L^\infty(\Omega), (dd^c v)^n \geq \mu, \limsup_{z \rightarrow \zeta} v(z) \leq f(\zeta), \forall \zeta \in \Omega\}$$

If we additionally assume that Ω is strictly pseudoconvex and μ is sufficiently regular (in particular any for any positive measure its canonical approximants are regular enough for this purpose), then a result from [Ce1] shows that actually this function solves the Dirichlet problem. In particular $(dd^c U(\mu, f))^n = \mu$.

Let us also recall some notions in the setting of compact Kähler manifold. Note that interactions between pluripotential theory, complex dynamics and differential geometry on such manifolds is a very timely topic and subject of intensive research. Hence we would like to show how our results can be used in this setting either.

Let X be compact n -dimensional Kähler manifold equipped with fundamental Kähler form ω (that is d -closed strictly positive globally defined form) given in local coordinates by

$$\omega = \frac{i}{2} \sum_{k,j=1}^n g_{k\bar{j}} dz^k \wedge d\bar{z}^j$$

We assume that the metric is normalized so that

$$\int_X \omega^n = 1.$$

Recall that

$$PSH(X, \omega) := \{\phi \in L^1(X, \omega) : dd^c \phi \geq -\omega, \phi \in \mathcal{C}^\dagger(X)\}$$

where $\mathcal{C}^\uparrow(X)$ denotes the space of upper semicontinuous functions. We call the functions that belong to $PSH(X, \omega)$ ω -plurisubharmonic (ω -psh for short). A very similar notion of *admissible* functions (additionally \mathcal{C}^∞ smoothness and strict inequality in definition of $PSH(X, \omega)$ are imposed) is widely used in differential geometry and has a much longer history.

One would like also to define the Monge-Ampère operator

$$(\omega_u)^n := (\omega + dd^c u)^n$$

in this setting. Since locally functions that belong to $PSH(X, \omega)$ are standard plurisubharmonic function minus a (smooth) potential for the form ω one can use classical results from pluripotential theory to define this for bounded u . This approach was used in [K1]. Recently in a series of articles Guedj and Zeriahi ([GZ1], [GZ2]) developed intrinsic pluripotential theory on compact Kähler manifolds. In particular in [GZ2] they defined Cegrell classes in this setting. We recall below the definition of the class $\mathcal{E}(X, \omega)$.

For every $u \in PSH(X, \omega)$ $(\omega + dd^c \max(u, -j))^n$ is a well defined probability measure regardless u is bounded or not. By [GZ2] the sequence of measures $\chi_{\{u > -j\}}(\omega + dd^c \max(u, -j))^n$ is always increasing and one defines

$$\mathcal{E}(X, \omega) := \{u \in PSH(X, \omega) \mid \lim_{j \rightarrow \infty} \int_X \chi_{\{u > -j\}}(\omega + dd^c \max(u, -j))^n = 1\}.$$

These functions might be unbounded, but the integral assumption ensures that there is no room for "bad things" to happen near $\{u = -\infty\}$. Then one defines

$$(\omega + dd^c u)^n := \lim_{j \rightarrow \infty} \chi_{\{u > -j\}}(\omega + dd^c \max(u, -j))^n.$$

In particular Monge-Ampère measures of functions from $\mathcal{E}(X, \omega)$ do not charge pluripolar sets. We refer to [GZ2] for a discussion of that notion.

3. PROOF OF THE MAIN THEOREM

Our idea will be quite similar in spirit to that in the proof of Theorem 1.2. We shall find appropriate sequences u_j , v_j for which Theorem 1.2 holds, and prove that they converge in a suitable way to u and v respectively ensuring the weak convergence of $(dd^c u_j)^k \wedge (dd^c v_j)^{n-k}$ towards $(dd^c u)^k \wedge (dd^c v)^{n-k}$. This is the point where we need the canonical approximants. The delicate point is that (by an example in [CK1]) the weak star convergence $(dd^c u_j)^n$ towards $(dd^c u)^n$ (even if all the functions considered have the same boundary values) does not imply strong enough convergence (that is convergence in capacity) of u_j towards u . So we have to use some special sequences of approximants. On the other hand, taking for example convolutions would be enough for the convergence in capacity but the inequality for the approximants is unclear.

First we consider the case of *bounded* u and v . In fact for most applications not requiring the theory of Cegrell classes this case is sufficient.

Now note that the claimed inequality is a local property, hence it suffices to prove it in a (small) ball \mathbb{B}^n , such that the functions u , v are defined in a neighbourhood of it. Let m_j , n_j be two sequences of smooth functions on $\partial \mathbb{B}^n$, decreasing to $u|_{\partial \mathbb{B}^n}$ and $v|_{\partial \mathbb{B}^n}$ respectively. Let u_j , v_j solve

$$\begin{cases} u_j \in PSH(\mathbb{B}^n) \cap L^\infty(\mathbb{B}^n) \\ (dd^c u_j)^n = (dd^c u)^n_j \\ u_j|_{\partial \mathbb{B}^n} = m_j \end{cases}$$

$$\begin{cases} v_j \in PSH(\mathbb{B}^n) \cap L^\infty(\mathbb{B}^n) \\ (dd^c v_j)^n = (dd^c v)_j^n \\ v_j|_{\partial \mathbb{B}^n} = n_j \end{cases}$$

We recall that $(dd^c u)_j^n$ is the canonical approximation of the measure $(dd^c u)^n$. By Theorem 2.2 such solutions exist.

Before we proceed we would like to point out some subtleties. If u and v were continuous, the sequences m_j, n_j would be redundant (since we can work with merely continuous boundary data as well). This point causes some technical problems in the proof. Also we need here to use the canonical approximants for the measures on the right hand side instead of the measures themselves, for the following reason: The Dirichlet problem

$$\begin{cases} u_j \in PSH(\mathbb{B}^n) \cap L^\infty(\mathbb{B}^n) \\ (dd^c u_j)^n = (dd^c u)^n \\ u_j|_{\partial \mathbb{B}^n} = m_j \end{cases}$$

need not have a solution continuous up to the boundary.

Proposition 3.1. *Let u_j, v_j be as above. Define $u := (\limsup_{j \rightarrow \infty} u_j)^*$, $v := (\limsup_{j \rightarrow \infty} v_j)^*$. Assume also that u_j (resp. v_j) tend to u (resp. v) in $L^1(d\lambda)$. Then we have*

$$(dd^c u_j)^k \wedge (dd^c v_j)^{n-k} \rightharpoonup (dd^c u)^k \wedge (dd^c v)^{n-k}, \quad \forall k \in \{1, \dots, n\}.$$

Proof. Note that the inequality

$$\int_{\Omega} (-\phi)^p d\mu \leq A(p) \left(\int_{\Omega} (-\phi)^p (dd^c \phi)^n \right)^{\frac{p}{n+p}}, \quad \phi \in \mathcal{E}_0(\Omega)$$

holds for any p with a constant $A(p)$ dependent on p , if μ is the Monge-Ampère measure of a bounded plurisubharmonic function (this follows easily from the theory in [Ce1]). So, by Theorem 2.2 and the discussion after it, we have (after passing to an appropriate subsequences, which for the sake of brevity, will also be denoted by u_j, v_j), that

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_{\Omega} |u - u_k| (dd^c u_k)^n &= 0, \\ \lim_{k \rightarrow \infty} \int_{\Omega} |v - v_k| (dd^c v_k)^n &= 0. \end{aligned}$$

Now Theorem 2.3 (see also the remark after it) give us that u_k , and v_k converge to u and v in capacity.

Now we are almost ready to approximate $(dd^c u)^k \wedge (dd^c v)^{n-k}$ by $(dd^c u_j)^k \wedge (dd^c v_j)^{n-k}$. Indeed a theorem of Xing (see [X1]) says that given a locally uniformly bounded sequences of PSH functions $\{w_k^j\}_{k=1}^\infty$, $j = 1, \dots, n$ converging in capacity to $w^j \in PSH$, the corresponding Monge-Ampère measures $dd^c w_k^1 \wedge \dots \wedge dd^c w_k^n$ converge weakly to $dd^c w^1 \wedge \dots \wedge dd^c w^n$.

The only thing that we (apriori) don't know, is whether the sequences u_j, v_j are locally uniformly bounded. However this difficulty can be bypassed by noticing that u_j, v_j are uniformly bounded in \mathcal{E}^1 norm (see [Ce1] or [K2]): to show this take any $U \Subset \Omega$ with

$\text{cap}(U, \mathbb{B}^n) < \epsilon$. Then

$$\begin{aligned}
& \int_U (dd^c u_j)^k \wedge (dd^c v_j)^{n-k} \leq \\
& \leq \int_{\mathbb{B}^n} -h_{U,\Omega} (dd^c(u_j + U(0, -m_j)))^k \wedge (dd^c(v_j + U(0, -n_j)))^{n-k} \leq \\
& \leq \left(\int_{\mathbb{B}^n} -(u_j + U(0, -m_j))(dd^c(u_j + U(0, -m_j)))^n \right)^{\frac{k}{n+1}} \times \\
& \times \left(\int_{\mathbb{B}^n} -(v_j + U(0, -n_j))(dd^c(v_j + U(0, -n_j)))^n \right)^{\frac{n-k}{n+1}} \left(\int_{\mathbb{B}^n} -h_{U,\Omega} (dd^c h_{U,\Omega})^n \right)^{\frac{1}{n+1}} \leq \\
& \leq C^{\frac{n}{n+1}} \text{cap}(U, \Omega)^{\frac{1}{n+1}} \leq C^{\frac{n}{n+1}} \epsilon^{\frac{1}{n+1}}
\end{aligned}$$

Where C is the uniform \mathcal{E}^1 bound for u_j , v_j , $h_{U,\Omega}$ is the relative extremal function of U and we have used the Hölder type inequalities, which is legal since $u_j + U(0, -m_j), v_j + U(0, -n_j)$ belong to \mathcal{E}_0 (see [Ce2]). The rigorous justification of the uniform \mathcal{E}^1 bound for the sequences is a bit technical and will be given in Lemma 3.2 below.

Now, again due to uniform \mathcal{E}^1 bounds, we have

$$\text{cap}(\{u_j < -s\}, \Omega) \leq \int_{\Omega} -\frac{|u_j|}{s} (dd^c h_{\{u_j < -s\}, \Omega})^n \leq \frac{C}{s}$$

with C independent of j and s (in fact much better estimates can be provided but these are satisfactory for our needs).

Fix s big enough such that $\text{cap}(\{u_j < -s\}, \Omega) \leq \epsilon, \forall j$. Then for any test function χ we have

$$\begin{aligned}
& \left| \int_{\mathbb{B}^n} \chi((dd^c u_j)^k \wedge (dd^c v_j)^{n-k} - (dd^c u)^k \wedge (dd^c v)^{n-k}) \right| \leq \\
& \leq \left| \int_{\{u_j \leq -s\} \cup \{v_j \leq -s\}} \chi((dd^c u_j)^k \wedge (dd^c v_j)^{n-k} - (dd^c u)^k \wedge (dd^c v)^{n-k}) \right| + \\
& + \left| \int_{\mathbb{B}^n} \chi((dd^c \max(u_j, -s))^k \wedge (dd^c \max(v_j, -s))^{n-k} - (dd^c u)^k \wedge (dd^c v)^{n-k}) \right|
\end{aligned}$$

But the first term is arbitrary small by the argument above and the second term tends to 0 due to Xing's theorem. So, we obtained the desired result. \square

Lemma 3.2. *There is an absolute constant C independent of j such that*

$$\int_{\mathbb{B}^n} -(v_j + U(0, -n_j))(dd^c(v_j + U(0, -n_j)))^n < C$$

Proof. Consider the function $g_j := U((dd^c v_j)^n, 0)$. From [Ce1] we know that $g_j \in \mathcal{E}_0$ and $(dd^c g_j)^n = (dd^c v_j)^n$. Hence by comparison principle applied to the pair $v_j, g_j + U(0, n_j)$ we get

$$v_j + U(0, -n_j) \geq g_j + U(0, n_j) + U(0, -n_j).$$

Let $h_j := U(0, n_j) + U(0, -n_j)$. By inequality from Proposition 2.1 we get

$$\int_{\mathbb{B}^n} -(v_j + U(0, -n_j))(dd^c(v_j + U(0, -n_j)))^n \leq \int_{\mathbb{B}^n} -(g_j + h_j)(dd^c(g_j + h_j))^n.$$

The last term can be decomposed into a sum of terms of the type

$$\binom{n}{m} \int_{\mathbb{B}^n} -(g_j + h_j)(dd^c g_j)^m \wedge (dd^c h_j)^{n-m}, \quad m \in \{0, \dots, n\}.$$

Again by Cegrell inequalities such terms are controlled from above by some product of $\int_{\mathbb{B}^n} -g_j(dd^c g_j)^n$ and $\int_{\mathbb{B}^n} -h_j(dd^c h_j)^n$. But h_j are uniformly bounded, while

$$\int_{\mathbb{B}^n} -g_j(dd^c g_j)^n = \int_{\mathbb{B}^n} -g_j(dd^c v_j)^n \leq \int_{\mathbb{B}^n} -(v_j + U(0, -n_j))(dd^c v_j)^n,$$

by the comparison principle. Now $U(0, -n_j)$ is uniformly bounded, $(dd^c v_j)^n$ have uniformly bounded total masses, and $\sup_j \int_{\mathbb{B}^n} -v_j(dd^c v_j)^n$ is finite by Theorem 2.2. Hence we have obtained the claimed uniform bound. \square

Now we prove our main inequality in this case:

Theorem 3.3. $u, v \in PSH(\Omega) \cap L^\infty(\Omega)$ satisfy

$$(dd^c u)^n \geq f d\mu, \quad (dd^c v)^n \geq g d\mu$$

where μ is an arbitrary positive measure and $f, g \in L^1(\Omega, d\mu)$, then

$$(dd^c u)^k \wedge (dd^c v)^{n-k} \geq f^{\frac{k}{n}} g^{\frac{n-k}{n}} d\mu$$

Proof. Consider the canonical approximation as in Theorem 3.1. We have that

$$\begin{aligned} (dd^c u)^k \wedge (dd^c v)^{n-k} &= \lim_{j \rightarrow \infty} (dd^c u_j)^k \wedge (dd^c v_j)^{n-k} \geq \\ &\limsup_{j \rightarrow \infty} \sum_j \chi_{I_k^j} \frac{(\int_{I_k^j} (dd^c u)^n)^{\frac{k}{n}} (\int_{I_k^j} (dd^c v)^n)^{\frac{n-k}{n}}}{dV(I_k^j)} dV \geq \\ &\geq \limsup_{j \rightarrow \infty} \sum_j \chi_{I_k^j} \frac{(\int_{I_k^j} f d\mu)^{\frac{k}{n}} (\int_{I_k^j} g d\mu)^{\frac{n-k}{n}}}{dV(I_k^j)} dV \geq \\ &\geq \limsup_{j \rightarrow \infty} \sum_j \chi_{I_k^j} \frac{(\int_{I_k^j} f^{\frac{k}{n}} g^{\frac{n-k}{n}} d\mu)}{dV(I_k^j)} dV = f^{\frac{k}{n}} g^{\frac{n-k}{n}} d\mu \end{aligned}$$

where we have used Theorem 1.2 and the Hölder inequality. \square

The case of n different functions instead of just two goes in the same way. The only difference is that we must use the generalised Hölder inequality (for n functions) instead of the classical one that we used above.

This result can be generalised to unbounded plurisubharmonic functions. We show below that our inequality remains true provided μ does not charge pluripolar sets. As the example in the next section shows the result is sharp. Since this is a purely local result we state it in terms of the Cegrell classes in a hyperconvex domain.

Theorem 3.4. $u, v \in PSH(\Omega) \cap \mathcal{E}(\Omega)$ satisfy

$$(dd^c u)^n \geq f d\mu, \quad (dd^c v)^n \geq g d\mu$$

assume moreover that μ does not charge pluripolar sets. Then we have the same conclusion as in the above theorem.

Proof. We recall the following known inequality which is a special case of Demailly's inequality (see for example [KH]):

$$(3.1) \quad (dd^c \max(u, -j))^n \geq \chi_{\{u > -j\}} (dd^c u)^n$$

for every u in $\mathcal{E}(\Omega)$. By the monotone convergence and the result in the bounded case we obtain

$$\begin{aligned} (dd^c u)^k \wedge (dd^c v)^{n-k} &= \lim_{j \rightarrow \infty} (dd^c \max(u, -j))^k \wedge (dd^c \max(v, -j))^{n-k} \geq \\ &\geq \limsup_{j \rightarrow \infty} (\chi_{\{u > -j\}} f)^{\frac{k}{n}} (\chi_{\{v > -j\}} g)^{\frac{n-k}{n}} \mu \end{aligned}$$

the last term converges to $f^{\frac{k}{n}} g^{\frac{n-k}{n}} \mu$ (because μ does not charge the pluripolar set $\{u = -\infty\} \cup \{v = -\infty\}$), which proves the claim. \square

4. A COUNTEREXAMPLE

All the results above suggest that the inequalities might be true for every measure μ . Below we shall present a counterexample which shows that this is not the case. Of course such a measure must charge some pluripolar set.

This example is borrowed from Wiklund's paper [W2], where these functions were used in a different context:

Example 4.1. *Let*

$u_k = \max\{\frac{1}{k} \log |z_1|, k^2 \log |z_2|\}$, $v_k = \max\{\frac{1}{k} \log |z_2|, k^2 \log |z_1|\}$. *Then*

$$(dd^c u_k)^2 = (2\pi)^2 \frac{k}{2} \delta_0, \quad (dd^c v_k)^2 = (2\pi)^2 \frac{k}{2} \delta_0$$

but

$$\begin{aligned} dd^c u_k \wedge dd^c v_k &= (2\pi)^2 \frac{1}{2k^2} \delta_0 \\ (dd^c(u_k + v_k))^2 &= (2\pi)^2 (k + \frac{1}{k}) \delta_0 \end{aligned}$$

where δ_0 is the Dirac delta. In particular inequalities (1.3) and (1.4) both fail in this case. One can also show that Demailly inequality fails for these functions, so the result in [KH] is sharp too.

Proof. First let us compute $(dd^c u_k)^2$. This is already known (see [W2]) but first, we would like to outline the proof and secondly, we shall need that method later. Note that our function is pluriharmonic except on the (real) hypersurface $|z_1| = |z_2|^{k^3}$. It is enough to compute $(dd^c u_{k,j})^2$ with $u_{k,j} := \max\{u_k, -j\}$, since by a result of Błocki such a sequence of measures weakly converge to $(dd^c u_k)^2$ for this particular choice of u_k (because this function belongs to $W_{loc}^{1,2}$, or equivalently it belongs to the Cegrell class \mathcal{E} , see [Bl1], [Ce1]). Now proceeding as in [Bl1] we use the change of the variable

$$(x, y) \longrightarrow (\log |z_1|, \log |z_2|)$$

to confirm that

$$\int_{\mathbb{D}^2} (dd^c u_{k,j})^2 = \int_{x \leq 0, y \leq 0} MA(\bar{u}_{k,j})$$

where MA is the real Monge-Ampère operator and $\bar{u}_{k,j}(x, y) := \max\{\frac{1}{k}x, k^2y, -j\}$. By Alexandrov's theorem (see [Al]) the latter integral is equal to the volume of the gradient image, i.e.

$$\int_{x \leq 0, y \leq 0} MA(\bar{u}_{k,j}) = \lambda(\nabla \bar{u}_{k,j}(\{x \leq 0, y \leq 0\})), \quad \nabla \bar{u}_{k,j}(E) := \cup_{w \in E} \nabla \bar{u}_{k,j}(w)$$

where $\nabla \bar{u}_{k,j}(w) := \{t \in \mathbb{R}^n \mid u_{k,j}(w) + \langle s - w, t \rangle \leq u_{k,j}(s), \forall s \in \text{Dom } u_{k,j}\}$.

At points where $u_{k,j}$ is smooth $\nabla \bar{u}_{k,j}(w)$ is a singleton set consisting of the usual gradient of $u_{k,j}$, while at non-smooth points usually $\nabla \bar{u}_{k,j}$ is not a singleton. Hence at points where $u_{k,j}$ is smooth and equal to $\frac{1}{k}x$ we get that $\nabla \bar{u}_{k,j}(w) = \{(\frac{1}{k}, 0)\}$. Analogously

in the two other smooth regions $u_{k,j} = k^2 y$ and $u_{k,j} = -j$ we get that $\nabla \overline{u}_{k,j}(w)$ is equal to $\{(0, k^2)\}$ and $\{(0, 0)\}$, respectively. Note that the Lebesgue measure of the gradient image for this set is 0. Let now w is a point where (for example) $\frac{1}{k}x = k^2 y > -j$. Then one easily computes the gradient image to be the line segment joining $(\frac{1}{k}, 0)$ and $(0, k^2)$. Analogously for the other points where two of the three functions considered in the maximum coincide the gradient image is a line segment joining the corresponding endpoints. Finally at the point $(-kj, \frac{-j}{k^3})$ (all three functions coincide), the gradient image will be the full triangle with vertices $(\frac{1}{k}, 0)$, $(0, k^2)$, $(0, 0)$.

The analysis above shows us that the total mass of $(dd^c u_{k,j})^2$ over the unit bidisc is equal to $(2\pi)^2 \frac{k}{2}$. Also if we fix a set $U \subset \mathbb{D}^2$ disjoint from the origin, its logarithmic image would not contain $(-kj, \frac{-j}{k^3})$ for j large, hence the gradient image of that set will have zero Lebesgue measure. This shows that $(dd^c u_k)^2$ is concentrated at the origin and since we know the total mass we find that $(dd^c u_k)^2 = (2\pi)^2 \frac{k}{2} \delta_0$. Note that $(dd^c v_k)^2 = (2\pi)^2 \frac{k}{2} \delta_0$ by symmetry.

What is left is to compute $dd^c u_k \wedge dd^c v_k$. But note that

$$2dd^c u_k \wedge dd^c v_k = (dd^c(u_k + v_k))^2 - (dd^c u_k)^2 - (dd^c v_k)^2$$

so all we need to do is to compute $(dd^c(u_k + v_k))^2$. Note that

$$u_k + v_k = \max\{(k^2 + \frac{1}{k}) \log |z_1|, \frac{1}{k} \log |z_1 z_2|, (k^2 + \frac{1}{k}) \log |z_1|\}.$$

Arguing in the same way the gradient image of $\max\{\overline{u_k + v_k}, -j\}$ is the (obtuse) rectangle with vertices (clockwise) $(0, k^2 + \frac{1}{k})$, $(\frac{1}{k}, \frac{1}{k})$, $(k^2 + \frac{1}{k}, 0)$, $(0, 0)$ which has volume $k + \frac{1}{k^2}$. Hence as above $(dd^c(u_k + v_k))^2 = (2\pi)^2(k + \frac{1}{k^2})\delta_0$, and finally $dd^c u_k \wedge dd^c v_k = (2\pi)^2 \frac{1}{2k^2} \delta_0$. \square

5. THE KÄHLER MANIFOLD CASE

In this section we prove analogous inequalities on a compact Kähler manifold.

Theorem 5.1. *Let $u, v \in \mathcal{E}(X, \omega)$ be ω -plurisubharmonic functions on X , and let μ be a positive measure that does not charge pluripolar sets and $f, g \in L^1(d\mu)$. If*

$$(\omega + dd^c u)^n \geq f d\mu, \quad (\omega + dd^c v)^n \geq g d\mu$$

as measures, then

$$(\omega + dd^c u)^k \wedge (\omega + dd^c v)^{n-k} \geq f^{\frac{k}{n}} g^{\frac{n-k}{n}} d\mu, \quad \forall k \in \{1, \dots, n-1\}$$

Proof. By the definition of $\mathcal{E}(X, \omega)$ it is enough to work with bounded approximants (on the set $\{u > -j\} \cap \{v > -j\}$). Also since again the problem is local it is enough to prove such an inequality in a ball contained in a coordinate chart. But then it is equivalent to the result we have already obtained. \square

Finally we would like to mention a slight generalisation of the result above. Namely it is redundant to impose that both functions belong to the same space $\mathcal{E}(X, \omega)$. We have the following proposition:

Proposition 5.2. *Let $u \in \mathcal{E}(X, \alpha), v \in \mathcal{E}(X, \beta)$ be ω -plurisubharmonic functions on X (α and β are possibly different Kähler forms on X), μ be a positive measure that does not charge pluripolar sets and $f, g \in L^1(d\mu)$. If*

$$(\alpha + dd^c u)^n \geq f d\mu, \quad (\beta + dd^c v)^n \geq g d\mu$$

as measures, then

$$(\alpha + dd^c u)^k \wedge (\beta + dd^c v)^{n-k} \geq f^{\frac{k}{n}} g^{\frac{n-k}{n}} d\mu, \quad \forall k \in \{1, \dots, n-1\}$$

Proof. The proof above applies, since the argument for the transition from a ball in a chart to a ball in \mathbb{C}^n is independent of the potential for the Kähler form. \square

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